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A remark on alpha vacua for quantum field theories on de Sitter space

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Abstract: It is shown that the so-called α-vacua which have been proposed as candidates for states of free quantum fields on de Sitter space have infinitely strong fluctuations for typical observables as the averaged renormalized energy momentum tensor.

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Contents

1. Introduction

One of the big problems of quantum field theory on curved spacetime is the choice of a state which describes a given physical situation. In particular, on a generic (time dependent) spacetime, one cannot single out a state that can be viewed as a vacuum state in any natural way. There is, however, strong evidence that, in the case of free fields, physical states should satisfy the so called *Hadamard condition* [2, 11] which specifies the singularity structure at coinciding points[1] but leaves a lot of freedom at noncoinciding points. One thus obtains a whole class of admissible states, and it was shown, using methods from microlocal analysis and concepts from algebraic quantum field theory, that a full fledged construction of renormalized perturbative quantum field theory on generic backgrounds is possible for such states (see [4, 10] and references cited there).

On specific spacetimes, one may invoke other criteria to single out states. For example, one may ask for states which are invariant under the isometries of the spacetime (if any). For instance, on de Sitter space, Gibbons and Hawking [9] described an invariant state ω_0 which is obtained by a Wick rotation of an invariant state of the corresponding Euclidean theory.

Now, for free fields on a spacetime with a PCT symmetry Θ (like de Sitter space where Θ maps a points x to its antipodal point x_A) it is possible to obtain from a given representation of the quantum fields a new representation by applying a so-called "Bogoliubov automorphism." For a free hermitian Klein-Gordon field φ, this is defined by

$$\chi_\alpha : \varphi(x) \to \varphi_\alpha(x) \equiv \cosh\alpha\, \varphi(x) + \sinh\alpha\, \varphi(\Theta x)\,.$$

[1]It also specifies the singularity structure for lightlike related points that are not necessarily close to each other.

The field φ_α is again a solution to the Klein-Gordon equation, and it has the same commutator as φ,

$$[\varphi(x), \varphi(y)] = i\Delta(x, y) \tag{1.1}$$

where Δ is the difference between the retarded and the advanced Green's function of the Klein Gordon equation. Namely

$$\begin{aligned}[\varphi_\alpha(x), \varphi_\alpha(y)] &= i\cosh^2\alpha\,\Delta(x, y) + i\sinh^2\alpha\,\Delta(\Theta x, \Theta y) + \\ &\quad +i\sinh\alpha\cosh\alpha\,(\Delta(x, \Theta y) + \Delta(\Theta x, y)) \\ &= i\Delta(x, y)\,,\end{aligned}$$

where we have used in the last line that the PCT transformation Θ interchanges the retarded and the advanced propagator, so that $\Delta(\Theta x, \Theta y) = -\Delta(x, y)$, and where we have used $\Theta^2 = \mathrm{id}$.

The Bogoliubov automorphism χ_α commutes with the action of the de Sitter group. Thus, starting from the Gibbons-Hawking state ω_0 one obtains a 1-parameter family of invariant states,[2] the *alpha vacua* ω_α, which are defined in terms of their n-point functions by

$$\langle\varphi(x_1)\cdots\varphi(x_n)\rangle_{\omega_\alpha} = \langle\varphi_\alpha(x_1)\cdots\varphi_\alpha(x_n)\rangle_{\omega_0}\,. \tag{1.2}$$

But whereas the Gibbons-Hawking state satisfies the Hadamard condition, this is no longer true for the alpha vacua for $\alpha \neq 0$ [1].

This can be easily seen in terms of the symmetric part $G_\alpha(x, y)$ of the 2-point function (the antisymmetric part coincides with the commutator and is thus independent of the state). We use the fact that the symmetric part $G_0(x, y)$ of the 2-point function of ω_0 is invariant under Θ. We thus find

$$G_\alpha(x, y) = \cosh 2\alpha\, G_0(x, y) + \sinh 2\alpha\, G_0(x, \Theta y)\,. \tag{1.3}$$

Since G_0 is singular at coinciding points, it follows that G_α is singular also at antipodal points. But two Hadamard 2-point functions can differ at most by a smooth function, and because of the hyperbolic prefactors in eq. (1.3), none of the alpha vacua for $\alpha \neq 0$ can be a Hadamard state.

2. Fluctuations

We now want to discuss the consequences of the last fact. As a simple case we investigate the quantum field which corresponds to φ^2 in the classical theory. The case of energy-momentum tensor, or other field monomials, can be treated along similar lines.

It is well known (see, e.g., [5]) that in order to define φ^2 one has to subtract a divergent term, for instance the expectation value in the Hadamard state ω_0,

$$:\varphi^2(x):_0 = \lim_{y\to x}\, \varphi(x)\varphi(y) - \langle\varphi(x)\varphi(y)\rangle_{\omega_0}\,. \tag{2.1}$$

[2] As shown in [1], this family in fact encompasses all de Sitter invariant states.

The definition depends on the choice of the state used in the subtraction, but the use of a different Hadamard state leads only to a finite correction. One can even show that there is a state independent subtraction which depends only on the spacetime metric and its curvature at the point x such that the expectation values in all Hadamard states are finite. Moreover, averaging the field $:\varphi^2:_0$ against a smooth real valued test function f of compact support, one obtains a well defined operator in the quantum theory, with well defined n-point functions in any Hadamard state. In particular, this operator has finite fluctuations in such states.

The definition leads, however, to a divergent expectation value in an alpha vacuum because of the divergence of G_0 at coinciding points. One might try another subtraction method where instead the expectation value in an alpha vacuum is subtracted. The quantities $:\varphi^2:_\alpha$ then have well defined expectation values in all states in the Fock space built on the same alpha vacuum, or, phrased differently, one may insert $:\varphi^2:_\alpha$ into an arbitrary n-point function of an alpha vacuum such as

$$\Big\langle \varphi(x_1)\cdots\varphi(x_n) :\varphi^2(y):_\alpha \Big\rangle_{\omega_\alpha}$$

and will obtain a well defined distribution in the variables $x_1, \ldots, x_n, y$. But if the averaged field is to be taken seriously as a quantum observable we should also investigate its fluctuations. This amounts to 2 insertions and will, as we shall see, lead to serious problems. We may for simplicity look at the fluctuations in the state ω_α. Since by construction the expectation value in this state vanishes, the fluctuations are given in terms of the expectation values of the square of the operator. Using the fact that the alpha vacuum is a Gaussian state, we have to compute the integral

$$\int dxdy\, f(x)f(y)\Big[\langle\varphi(x)\varphi(y)\rangle_{\omega_\alpha}\Big]^2 . \tag{2.2}$$

As it stands the expression is not well defined, since a distribution can in general not be squared. Since we wish to test the ultraviolet scale we may look at test functions that are supported in a patch that is small compared to the de Sitter scale. Then the leading term in the 2-point function of ω_0, in normal coordinates around some point in the support of f, is just the massless Minkowski space 2-point function

$$D_+(x,y) = (2\pi)^{-3}\int \frac{d^3\mathbf{k}}{2|\mathbf{k}|}e^{ik(x-y)}$$

where $k = (|\mathbf{k}|, \mathbf{k})$. The square of D_+ is actually well defined which is due to the fact that the convolution of its Fourier transform with itself involves only the integration over a compact region in momentum space. This is the reason why, in a Hadamard state the fluctuations are finite. In an alpha vacuum, however, one obtains also a mixed contribution where $D_+(x,y)$ is multiplied with $D_-(x,y) = D_+(y,x)$. Such a term necessarily diverges; namely

$$\int dxdy\, f(x)f(y)D_+(x,y)D_+(y,x) = \int \frac{d^3\mathbf{k}}{2|\mathbf{k}|}\frac{d^3\mathbf{p}}{2|\mathbf{p}|}|\widehat{f}(k-p)|^2 .$$

Since f, being a smooth function that vanishes outside a compact region, has a strongly decaying Fourier transform $\widehat{f}$, the integration over $\mathbf{k}$ is essentially restricted to a small neighbourhood of $\mathbf{p}$. It thus amounts essentially to replace $|\mathbf{p}|$ by $|\mathbf{k}|$, and we remain with the linearly diverging integral

$$\text{const} \int \frac{d^3\mathbf{k}}{2|\mathbf{k}|^2} \,.$$

The energy-momentum tensor case can be discussed similarly. The result is a fifth power diverging integral. We leave to the reader the easy details.

3. Conclusions

We have shown that typical observables have infinitely strong fluctuations in α-vacua for $\alpha \neq 0$. This holds in particular for the averaged energy-momentum tensor. The obtained result is valid already at the level of free field theory. It is in agreement with certain results in the existing literature, although it uses a much simpler derivation in that it does not rely either on perturbation theory [6, 8] or by coupling the theory to gravity [12]. Therefore, α-vacua ($\alpha \neq 0$) are not suited for a semiclassical treatment of "quantum gravity." It remains, however, the possibility to cut-off the mode expansion of the fields at very high energy (Planck length), which would result in a "better" short distance behavior of the two point function, although breaking de Sitter invariance at such scales. This idea seems to be of interest for the purpose of inflation and the issue of "trans-planckian" physics [7]. We shall discuss this additional possibility in a forthcoming contribution.

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